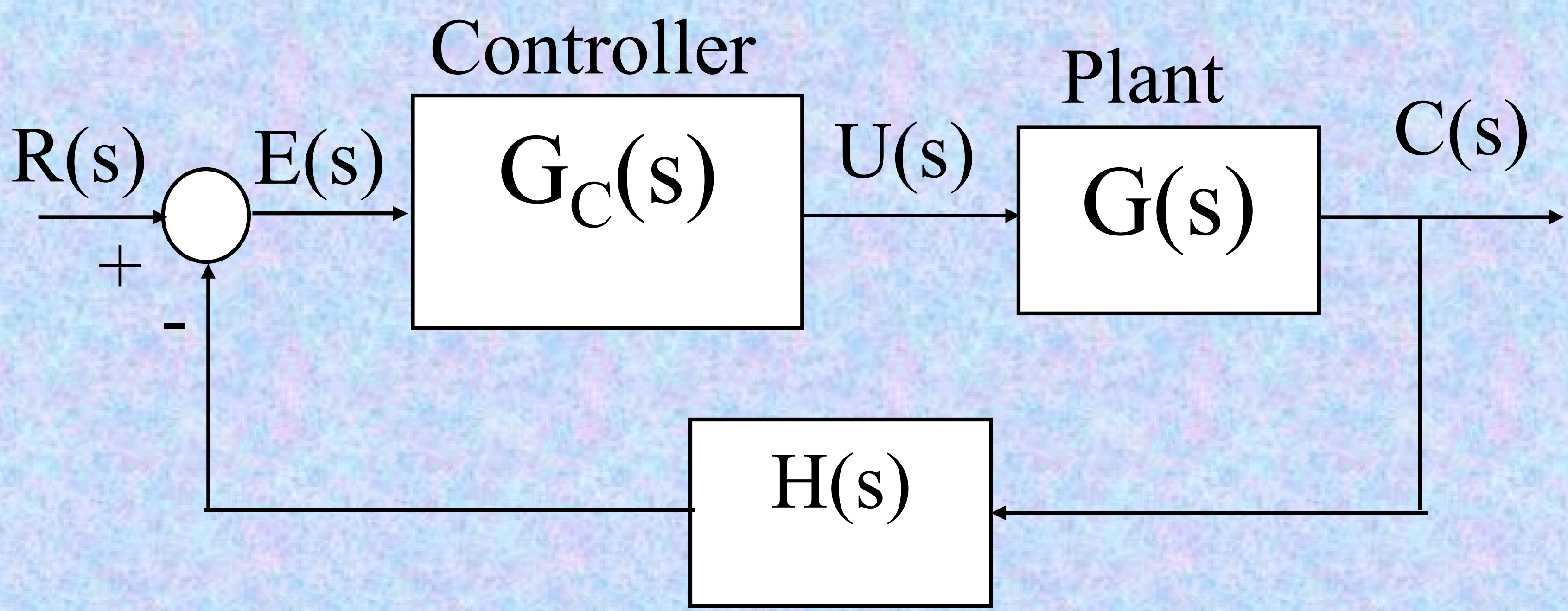
- السلام عليكم
- ورحمة الله
- وبركاته

Chapter 6

Root Locus Technique

•Introduction


Controller
Plant
R(s)
E(s)
$G_C(s)$
U(s)
G(s)
C(s)
+
-
H(s)

- **Stability**
- **The stability of the closed loop control system can be determined from the location of the**
- **closed loop poles (roots of the characteristic equation),**
- **weather the system is stable**
- **or unstable.**

Example:

Find the output response for a unit step input R(s)=1/s for the system has closed loop T.F:

$$\frac{C(s)}{R(s)} = \frac{10}{(s+2)(s+4)}$$

Solution

$$C(s) = \frac{10}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s+4)}$$

$$C(s) = \frac{1.25}{s} - \frac{2.5}{(s+2)} + \frac{1.25}{(s+4)}$$

$$c(t) = \underbrace{1.25}_{\text{Steady state}} \underbrace{-2.5e^{-2t} + 1.25e^{-4t}}_{\text{Transient}}$$

$$= C_{SS} + C_t(t)$$

- **As $t \rightarrow \infty$, $c_t(t) = 0$, transient output =0, and**
- **The system output, c(t) =**
- **steady state output, Css**

- **Such systems are called**
- ***absolutely stable systems***

Example 2:

Find the output response for a unit step input R(s)=1/s for the system has closed loop T.F:

$$\frac{C(s)}{R(s)} = \frac{10}{(s-2)(s+4)}$$

Solution

$$C(s)=\frac{10}{s(s-2)(s+4)}=\frac{A}{s}+\frac{B}{(s-2)}+\frac{C}{(s+4)}$$

$$C(s)=-\frac{1.25}{s}+\frac{0.833}{(s-2)}+\frac{0.416}{(s+4)}$$

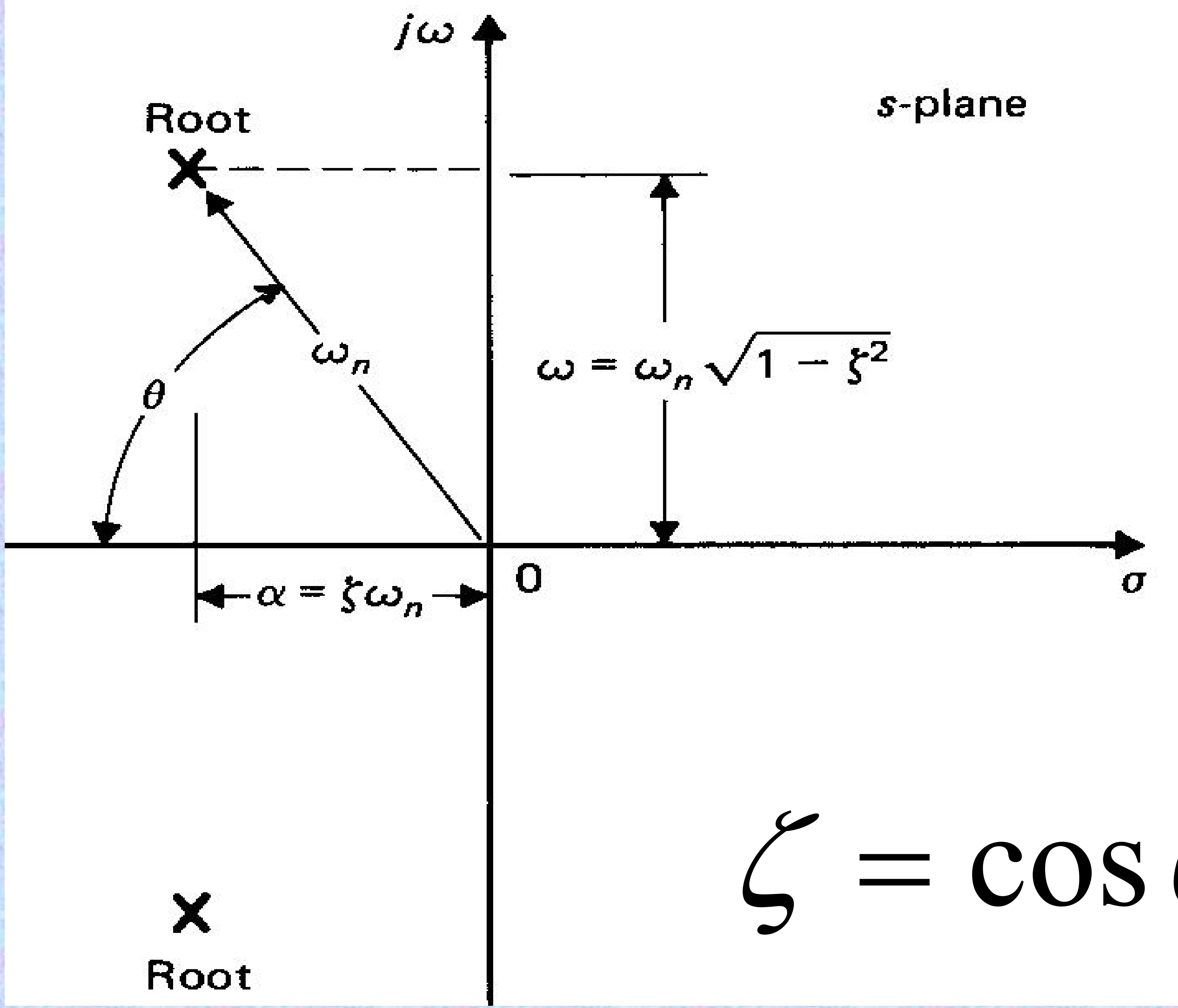
$$c(t)=-1.25+0.833\,e^{+2t}+0.416\,e^{-4t}$$

- As $t \to \infty$, $c(t) = \infty$, due to
- exponential term with positive
- index, transient go on increasing
- in amplitude.
- So such system is called unstable.

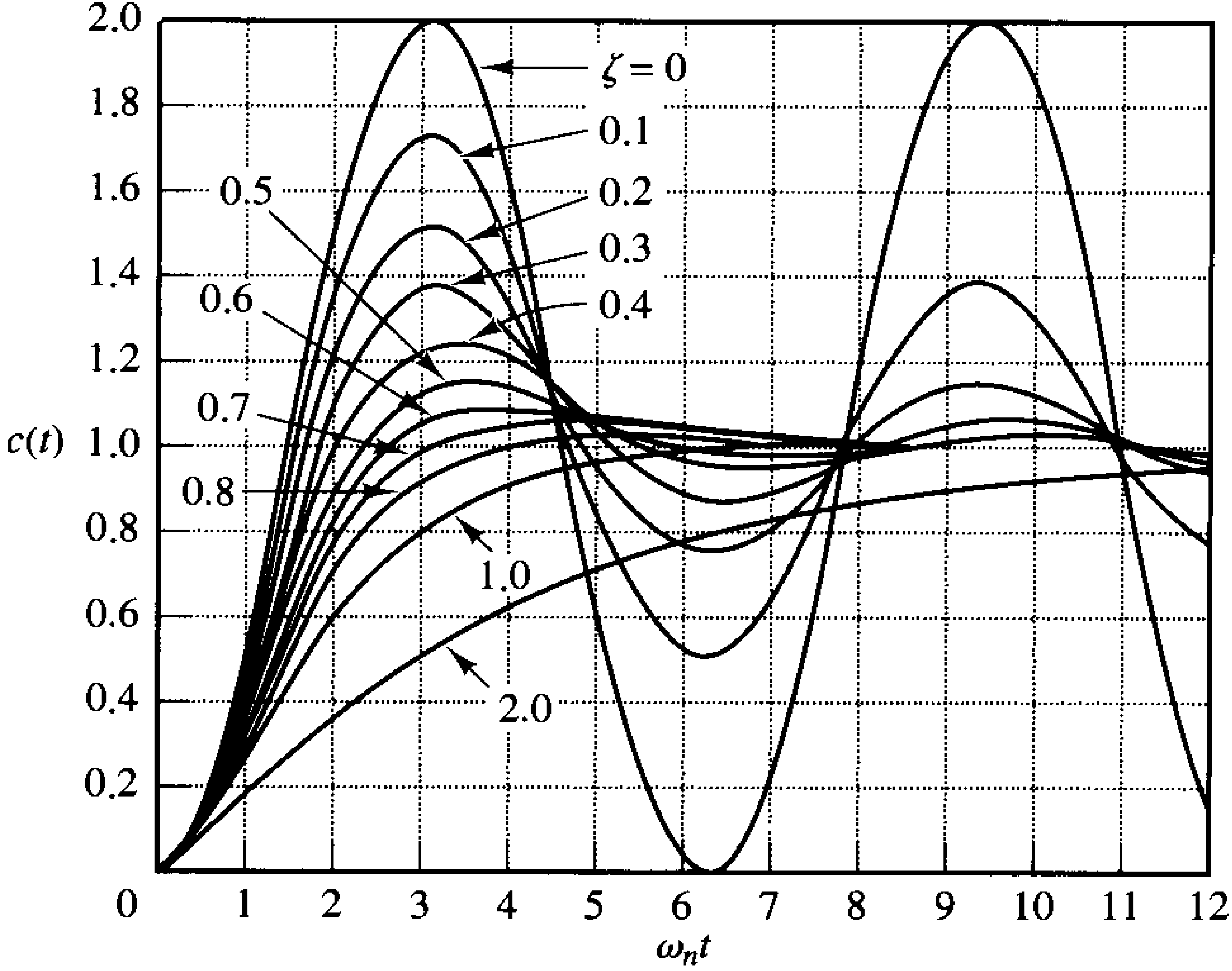
- **Stable Systems:**
- **If the closed loop poles of the system**
- **are located in the left half of s-plane,**
- **the system is *stable*.**
- **Unstable Systems:**
- **If any of the closed loop poles of the**
- **system are located in the right half of**
- **s-plane, the system is *unstable*.**

- **Relative Stability**
- **The system is relatively more**
- **stable or unstable depending on**
- **the settling time.**
- **The system is relatively more**
- **stable if settling time of the system**
- **is less than the other system.**
-

- <u>**Transient Response**</u>
- **The transient response of the**
- **system is related to the location**
- **of the closed loop poles of the**
- **system**



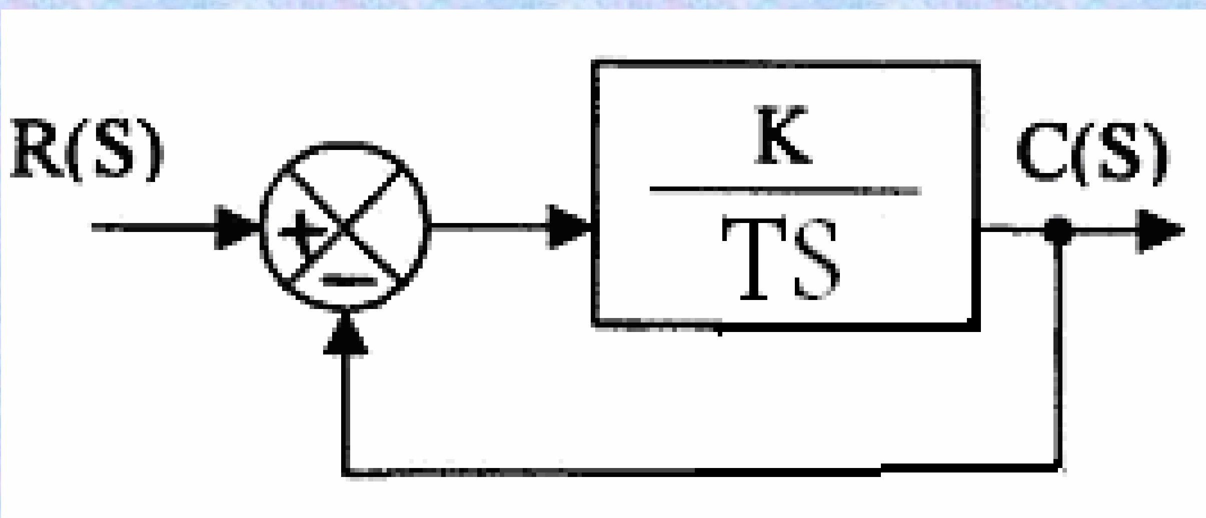
$$\zeta = \cos\theta$$



2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
1
2
3
4
5
6
7
8
9
10
11
12
$c(t)$
$\omega_n t$
$\zeta = 0$
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
1.0
2.0

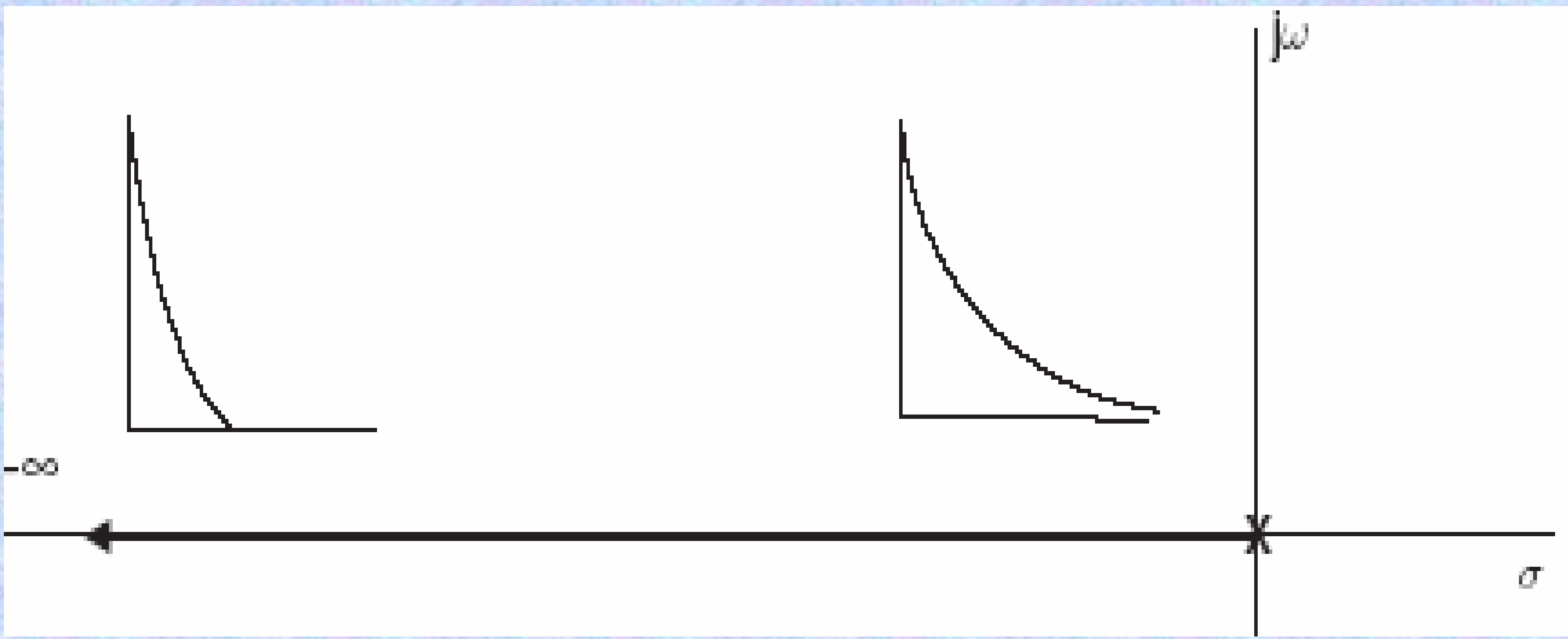
- **Root Locus Concept**
- **The root locus is the path of the**
- **roots of the characteristic**
- **equation in s-plane as system**
- **parameter (K) is changed.**
- **The root locus technique are**
- **used to study the changes in**
- **performance of linear systems that**
- **occur with variations of system**
- **parameters.**

- **Example 6.1:**
- **Construct the root locus**
- **diagram for the first order**
- **system shown in figure**

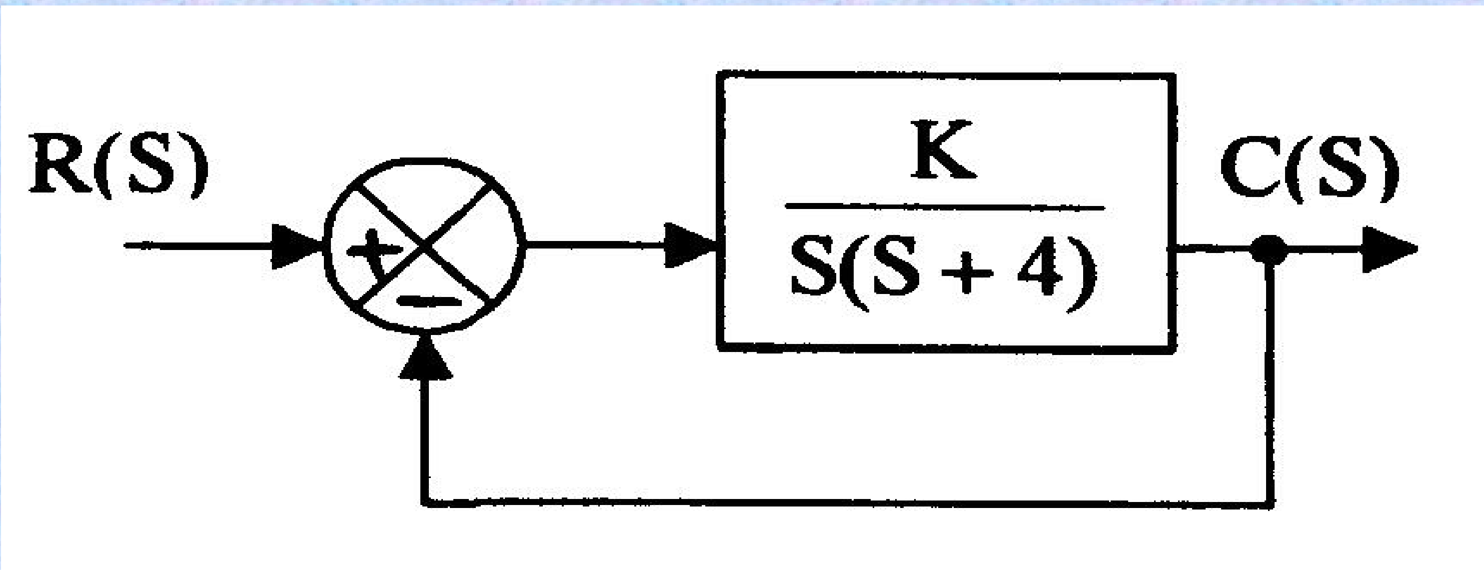


- **Solution:**
- Open loop transfer function is
- $G(s)H(s) = K / Ts$
- Characteristic equation is:
- $$1+G(s)H(s) = 0$$
- $$1+K / Ts =0$$
- $$Ts + K = 0$$
- Root of the characteristic equation is at $s = -K / T$

- When K is varied from zero to infinity
- the locus starts at the open loop
- pole s = 0 and terminates at minus
- infinity on the real axis as shown in
- figure 6.2



- **Example 6.2:**
- **Construct the root locus diagram**
- **for the second order system**
- **shown in figure 6.3.**



- **Solution:**
- The open loop transfer function is
- $G(s)H(s) = K / s(s+4)$
- The open-loop poles are:
- $s = 0, \quad s = -4$
- The characteristic equation is:
- $1+G(s)H(s)=0$
- $1 + K / s(s+4) = 0$
- $s^2 + 4s + K = 0$

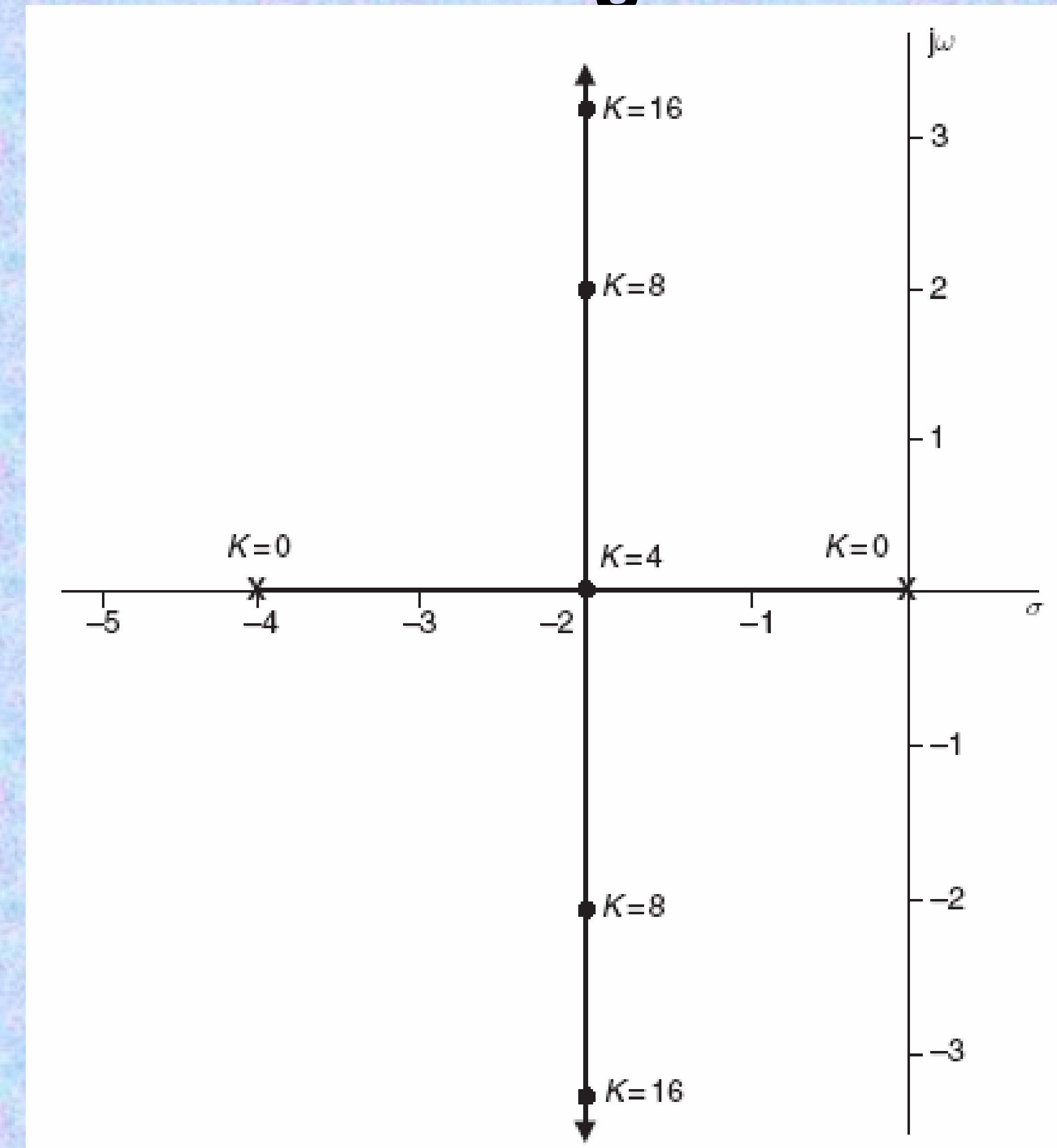
- **The roots are:**

$$\frac{-4 \pm \sqrt{16 - 4K}}{2} = -2 \pm \frac{\sqrt{(16 - 4K)}}{2}$$

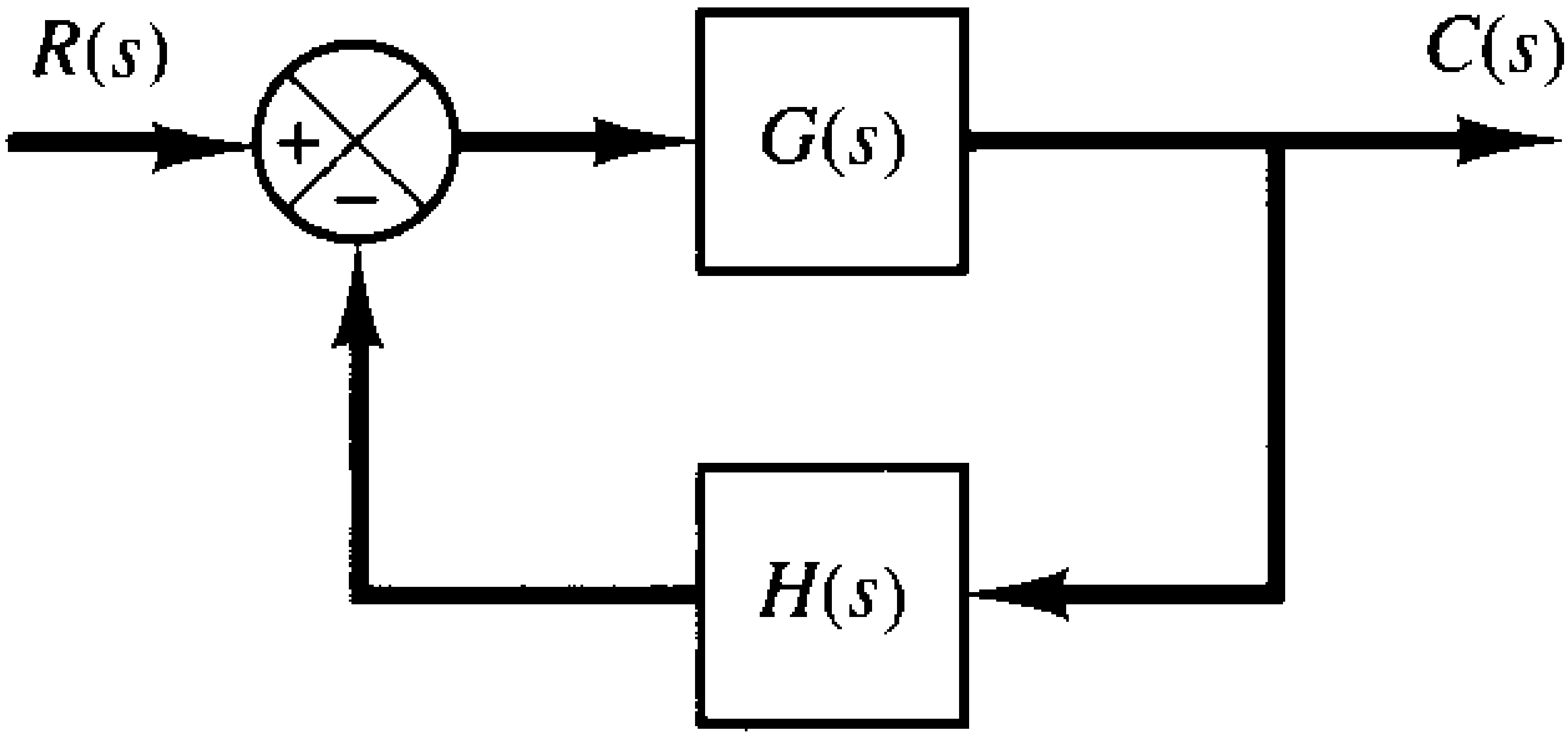
- **Table 6.1 shows the roots of the**
- **characteristic equation for**
- **different values of K.**

K	S1	S2
0	0	-4
4	-2	-2
8	$-2 + j2$	$-2 - j2$
16	$-2 + j3.46$	$-2 - j3.46$
10^6	$-2 + j1000$	$-2 - j1000$
:	:	:
∞	$-2+j\infty$	$-2-j\infty$

Figure 6.4 shows the corresponding root-locus diagram..



- **Root Locus of a Closed loop Control System**


R(s)
G(s)
C(s)
H(s)
+
−

- **The closed loop transfer**
- **function is given by:**

$$\frac{C(s)}{R(s)} = G_{CL}(s) = \frac{G(s)}{1 + G(s)H(s)}$$

- The characteristic equation of the
- closed loop systems is:

- $$1 + G(s)H(s) = 0 \qquad (6.2)$$

- $$G(s)H(s) = -1 \qquad (6.3)$$

- **Since equation (6.3) is a vector**
- **quantity, it can be represented in**
- **terms of magnitude and angle as:**
-

Angle Condition

$$\angle G(s)H(s) = \pm(2i+1)\pi \qquad \textbf{(6.4)}$$

Magnitude Condition

$$|G(s)H(s)| = 1 \qquad \textbf{(6.5)}$$

- **All the roots of the characteristic**
- **equation must satisfy the**
- **angle condition (Eq. 6.4) and magnitude condition (Eq. 6.5)**
-

The open loop transfer function $G(s)H(s)$ must be written in factored form as

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2).....(s+z_m)}{(s+p_1)(s+p_2)......(s+p_n)}$$

Applying the angle criteria, (equation (6.4)), we get

$$
\begin{aligned}
\angle G(s)H(s) &= \angle s+z_1 + \angle s+z_1 + \cdots + \angle s+z_m \\
&\quad - (\angle s+p_1 + \angle s+p_2 + \cdots + \angle s+p_n) \\
&= \pm(2i+1)\pi
\end{aligned}
$$

- i.e. for any point laying on a root locus of a system,
- the sum of the angles from the
- zeros to that point minus the sum
- of the angles from the poles to
- that point must satisfy the angle
- condition.

Applying the magnitude criteria

(equation (6.5)), we get:

$$\left|G(s)H(s)\right| = \frac{K\left|s+z_1\right|\left|s+z_2\right|\cdots\left|s+z_m\right|}{\left|s+p_1\right|\left|s+p_2\right|\cdots\left|s+p_n\right|} = 1$$

- **Example for angle condition**
- **Refer to example (6.2), the OLTF of a system is:**
- **$G(s)H(s)= K / s(s+4)$**
- **Find weather the points $s= -1$ and**
- **$s=-1+j2$ are on the root locus or**
- **not using angle condition.**

- **Solution:**
- **(a) Use angle condition for the point s=-1**

$$\angle G(s)H(s)\Big|_{\text{at } s=-1} = \pm(2i+1)\pi$$

$$\angle G(s)H(s)\Big|_{\text{at } s=-1} = \frac{\angle K}{\angle -1 + \angle 3} = \frac{0^\circ}{180^\circ + 0^\circ} = -180^\circ$$

- **Since s= -1 satisfy the angle condition, then it is on the root locus.**

- **(b) Use angle condition for the point s=-1+j2**

$$\angle G(s)H(s)\Big|_{\text{at } s = -1 + j2} = \frac{\angle K}{\angle(-1 + j2) + \angle(3 + j2)}$$

$$= \frac{0^\circ}{-63.43^\circ + 33.7^\circ} = -29.73^\circ$$

- **the point at s= -1+j2 is not on the root locus.**

- Example for magnitude condition
- Refer to example (6.2), the OLTF of a system is:
- $G(s)H(s)= K / s(s+4)$
- and the points $s = -1$ is on the root locus.
- Use the magnitude condition to
- find the value of K at this point.

- **Solution:**
- **Use magnitude condition for the point s=-1**

$$\left|G(s)H(s)\right|_{at\ s=-1} = \frac{|K|}{|s|\,|s+4|} = 1$$

$$\frac{|K|}{|-1|\,|3|} = 1 \qquad \therefore K = 3$$

- **Example 2 (Angle condition)**
- **Assume OLTF is:**
- **G(s)H(s)= K / s(s+2)(s+4)**
- **Find weather the points s=-0.75 and s=-1+j4 are on the root locus or not using angle condition.**

- **Solution:**
- **(a) Use angle condition for the point s=-0.75**

$$\angle G(s)H(s)\Big|_{\text{at s=-0.75}} = \pm(2i+1)\pi$$

$$\angle G(s)H(s)\Big|_{\text{at s=-0.75}} = \frac{\angle K}{\angle -0.75 + \angle 1.25 + \angle 3.25}$$

$$= \frac{0^\circ}{180^\circ + 0^\circ + 0^\circ} = -180^\circ$$

- **(b) Use angle condition for the point s=-1+j4**

$$\angle G(s)H(s)\Big|_{\text{at } s=-1+j4} = \frac{\angle K}{\angle(-1+j4)+\angle(1+j4)+\angle(3+j4)}$$

$$= \frac{0^\circ}{-75.96^\circ + 75.96^\circ + 53.13^\circ} = -53.13^\circ$$

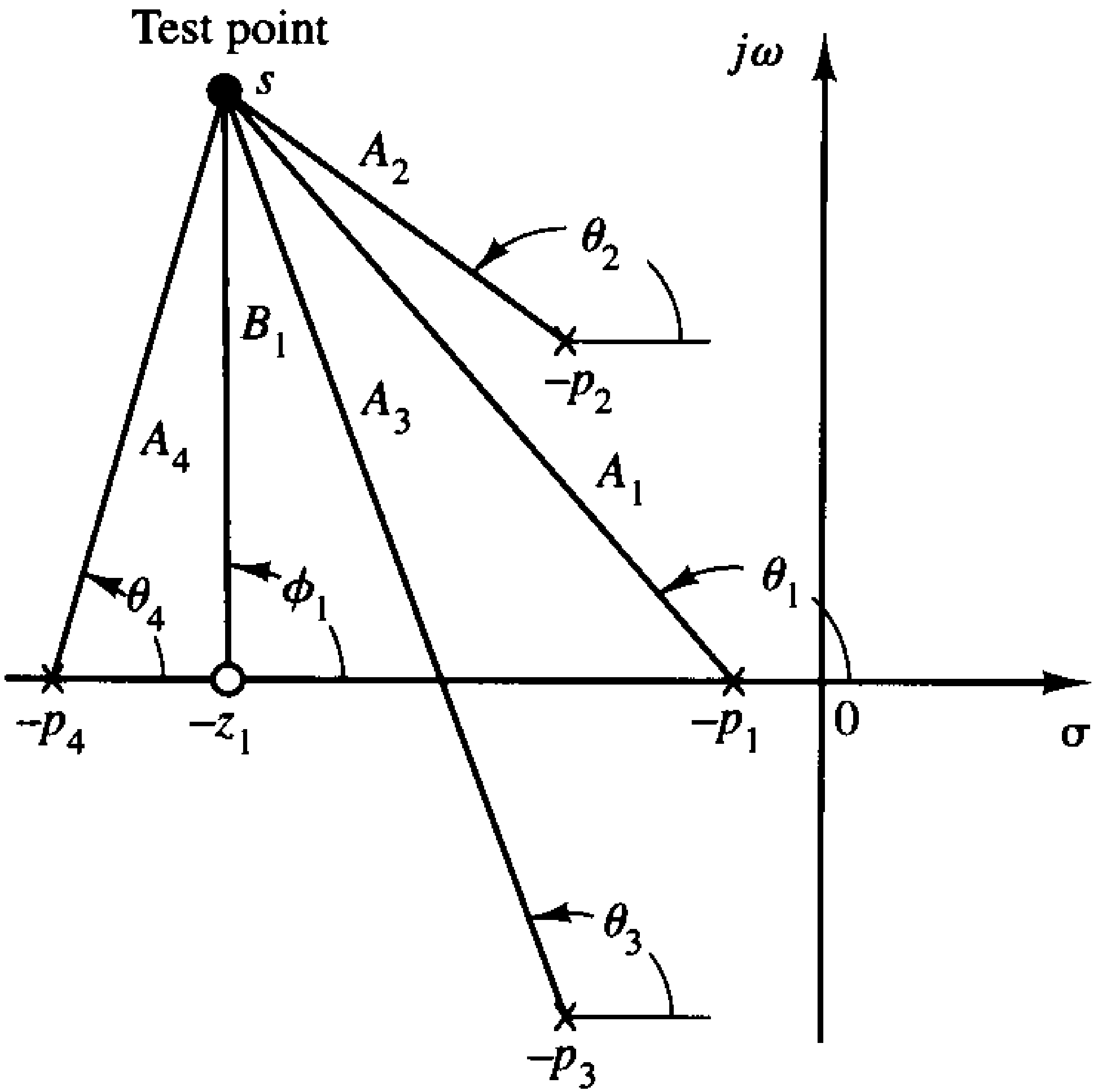
- **The point s= -1+j4 is not on the root locus**

- **Graphical Method of determining the value of K**
- **Example:**
- **If G(s)H(s) is given by**

$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

- **Where p1 and p4 are real poles while p2 and p3 are complex conjugate poles of G(s)H(s), as shown in figure**

- (a) Find weather the test point S is
- on the root locus or not using
- angle condition.
- (b) If it is on the root locus, find
- the value of K at this point.
- **Solution**
- (a) Join all open loop poles and
- zeros with the test point S as
- shown in figure.


Test point
s
jω
A2
θ2
B1
−p2
A3
A4
A1
θ1
θ4
ϕ1
−p4
−z1
−p1
0
σ
θ3
−p3

- **Then the angle of G(s)H(s) is**

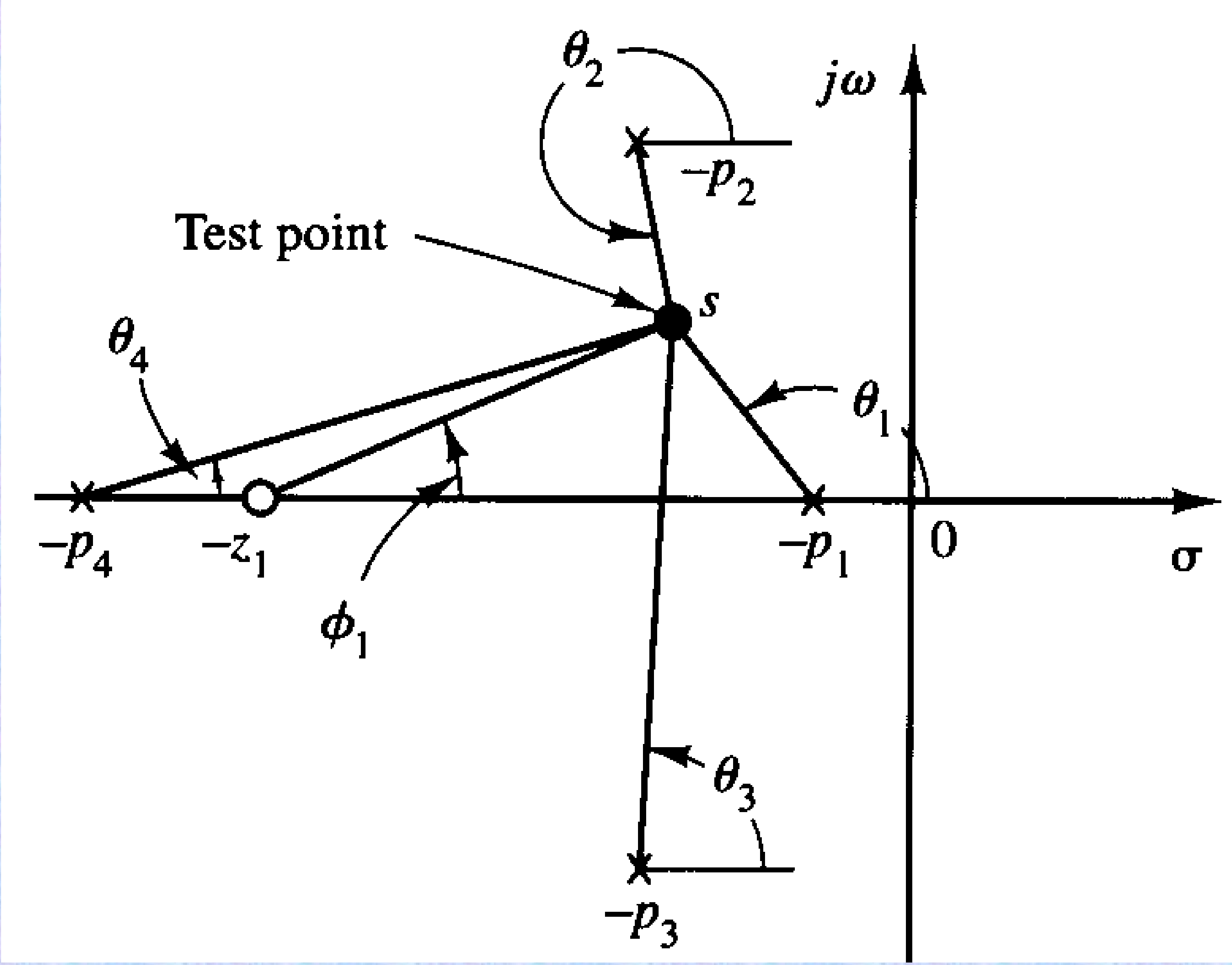
$$\angle G(s)H(s) = \angle s + z_1$$
$$- (\angle s + p_1 + \angle s + p_2 + \angle s + p_3 + \angle s + p_4)$$
$$= \phi_1 - (\theta_1 + \theta_2 + \theta_3 + \theta_4)$$

- **The magnitude of G(s)H(s) is:**

$$\left|G(s)H(s)\right| = \frac{K\left|s+z_1\right|}{\left|s+p_1\right|\left|s+p_2\right|\left|s+p_3\right|\left|s+p_4\right|}$$

$$= \frac{KB_1}{A_1A_2A_3A_4} = 1$$

$$K = \frac{A_1A_2A_3A_4}{B_1}$$


θ_2
$j\omega$
$-p_2$
Test point
s
θ_4
θ_1
$-p_4$
$-z_1$
ϕ_1
$-p_1$
0
σ
θ_3
$-p_3$

- **Example:**
- **Construct the root locus diagram**
- **for the system whose OLTF**
- **G(s)H(s) is:**

$$G(s)H(s) = \frac{K(s+1)}{s(s+5)}$$

- **Solution:**
- **The open-loop poles are at**
- **s = 0, s = -5**
- **The open-loop zero is at s = - 1**

- **The characteristic equation is:**

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s+5)} = 0$$

- **$s^2 + (5 + K)s + K = 0$**

- **The roots are:**

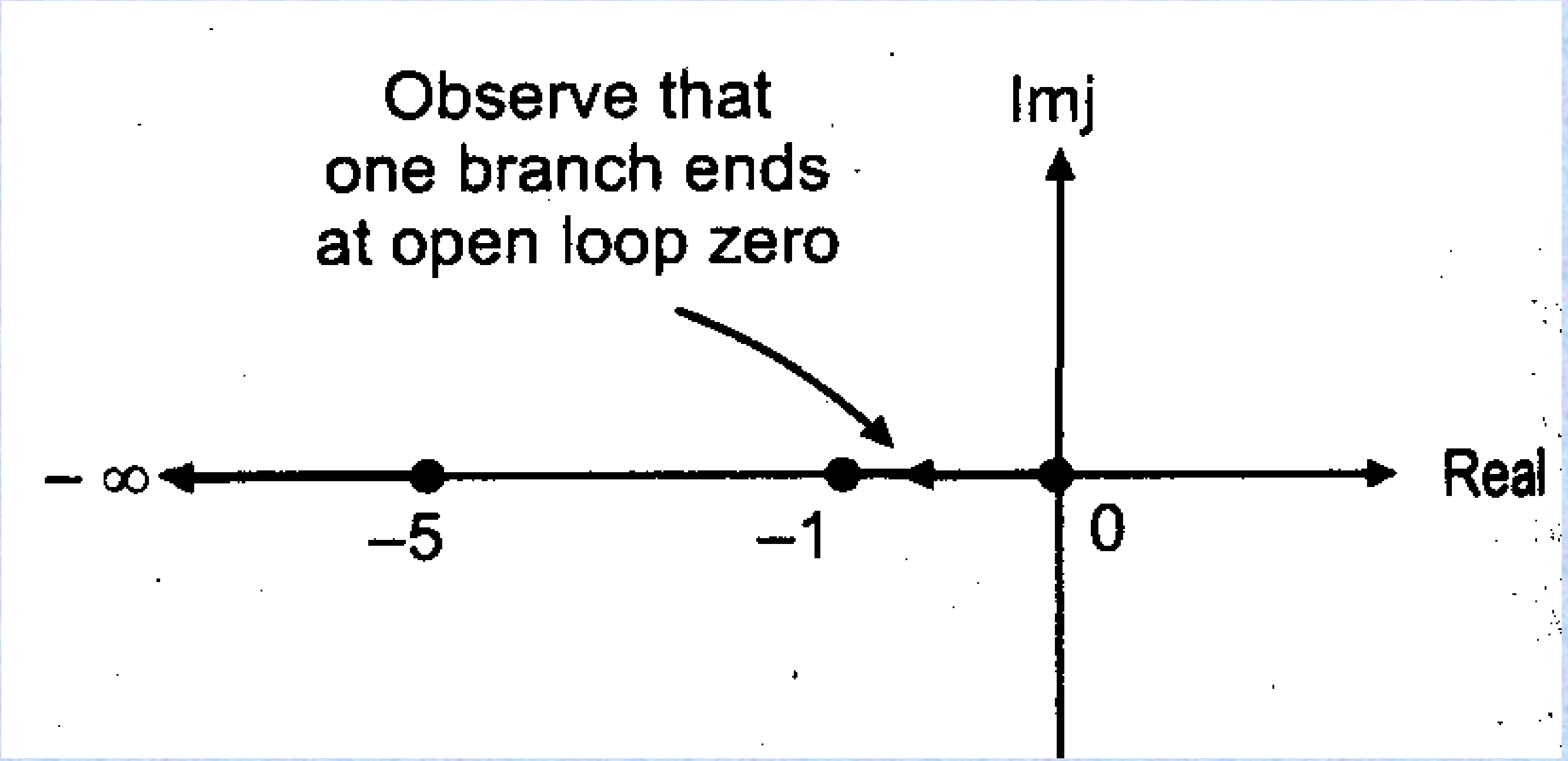
$$s_1, s_2 = \frac{-(K+5) \pm \sqrt{(K+5)^2 - 4K}}{2}$$

$$= \frac{-(K+5)}{2} \pm \frac{\sqrt{(K^2 + 6K + 25}}{2}$$

- **Table shows the roots of the characteristic equation for different values of K.**

K	S1	S2
0	0	-5
1	-0.172	-5.83
5	-0.53	-9.47
1000	-0.996	-1004
:	:	:
∞	-1	-∞

Figure shows the corresponding root-locus diagram .



- The root locus has two branches (number of open loop poles).
- Both branches are starting from $s=0$ and $s=-5$ which are open loop poles.
- One of the branches terminates at $s=-1$ which is open loop zero, while other branch is terminating at infinity.

- **Rules for Construction the Root Locus**
- **1● rearrange the open loop transfer**
- **function, if necessary, in the form of**

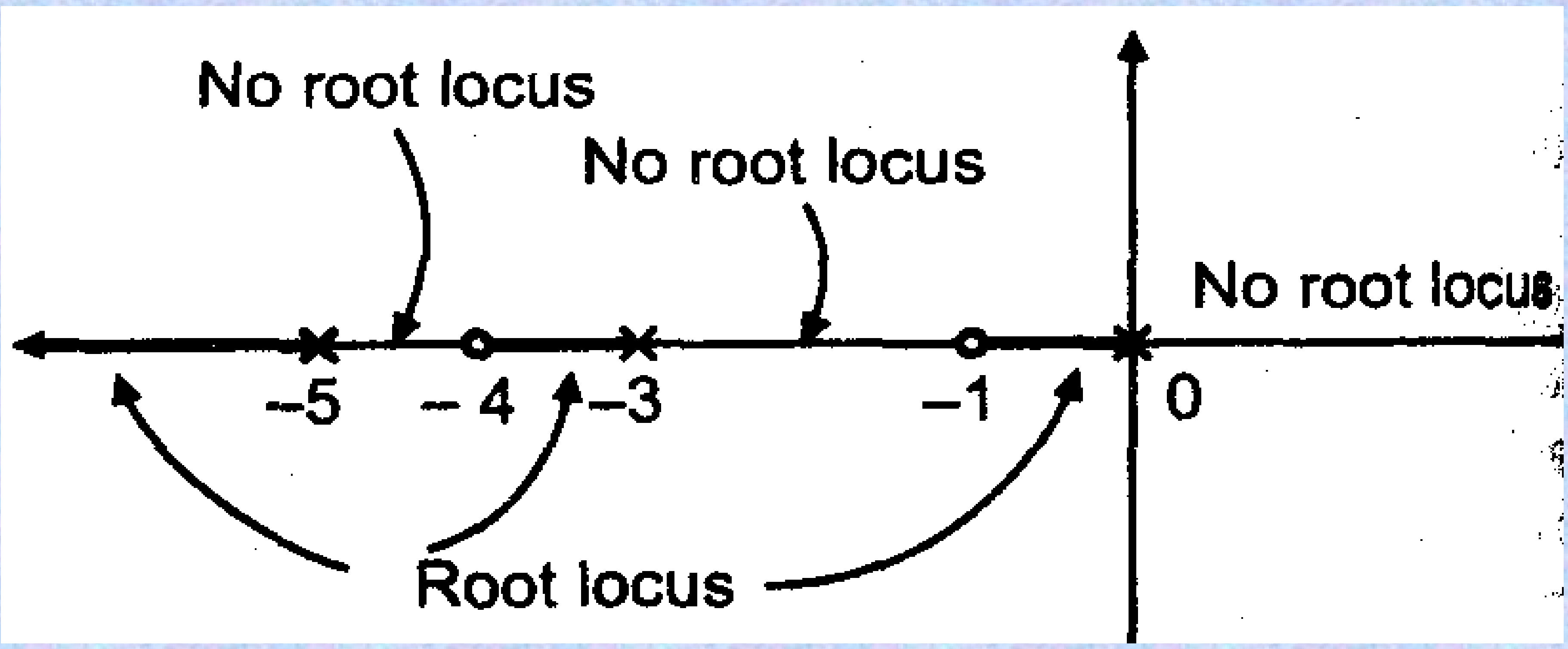
$$G(s)H(s) = \frac{K(s+z_1)(s+z_2).....(s+z_m)}{(s+p_1)(s+p_2)......(s+p_n)}$$

- **Locate the open loop poles and zeros**
- **on the s-plane with selected symbols**

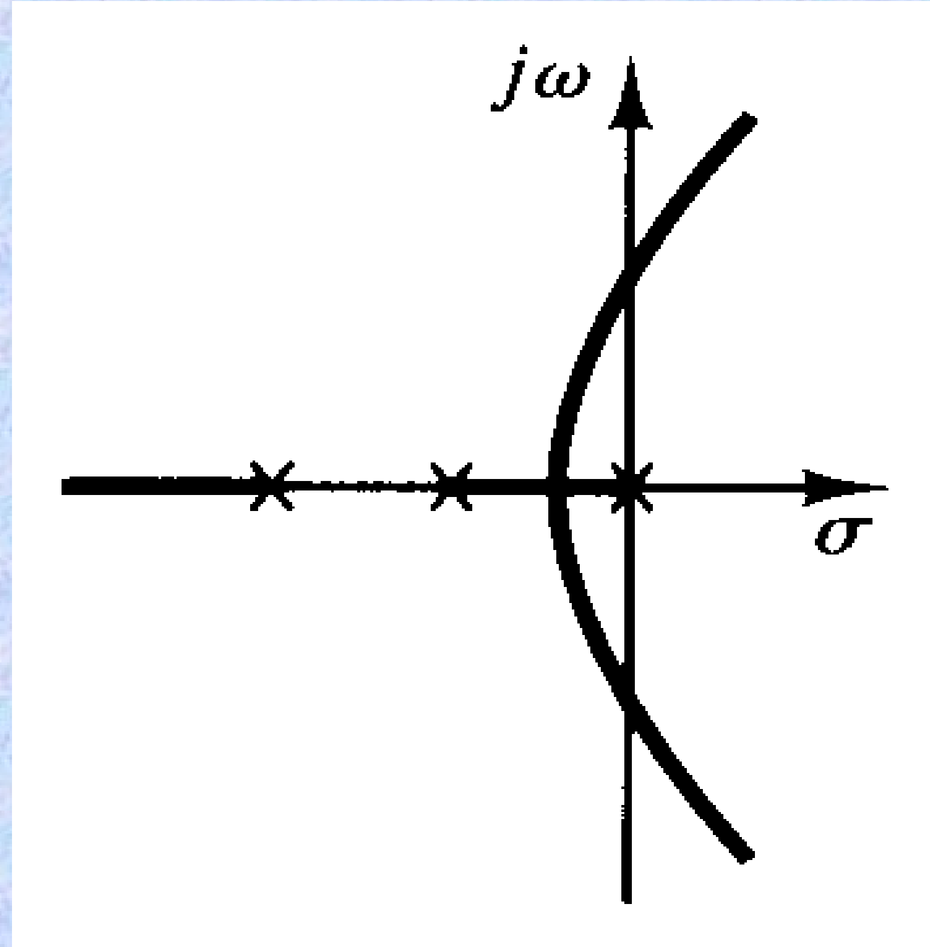
- **2. Branches of the root loci.**
- ● **The number of branches on the**
- **root loci is equal to the order of the system (number of poles of GH).**
- ● **The branches of the root loci are**
- **start at each of the poles of GH (k = 0) and terminates at the zeros of GH or at s=∞.**
- ● **the number of poles of G(s)H(s) ,**
- ***n* is more than or equal the number of zeros *m*.**

- **The number of branches terminating at s=∞ (zero at infinity) equals the number of open-loop poles minus zeros (*n-m*).**

- 3● Root locus on the real axis
- The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros as shown in the following figure.

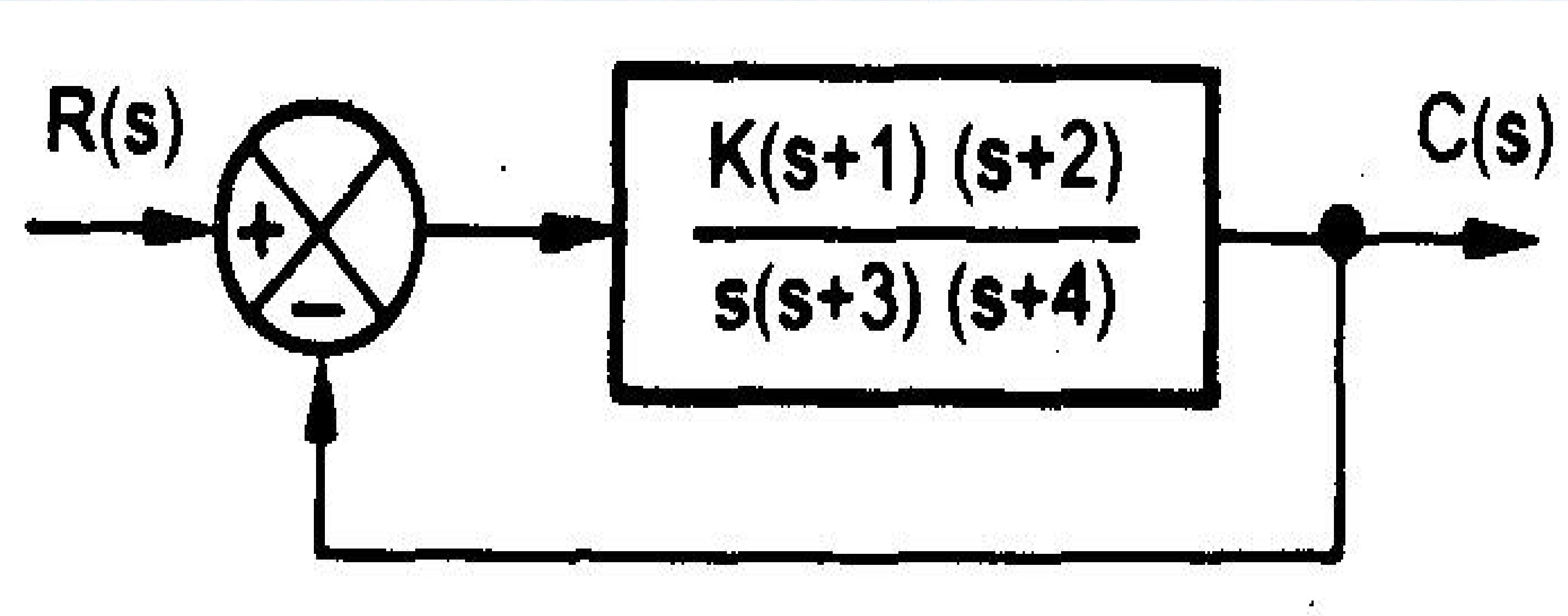

No root locus
No root locus
No root locus
–5
– 4
–3
–1
0
Root locus

- •The root locus are symmetrical with respect to the real axis (horizontal axis) of the s-plane .

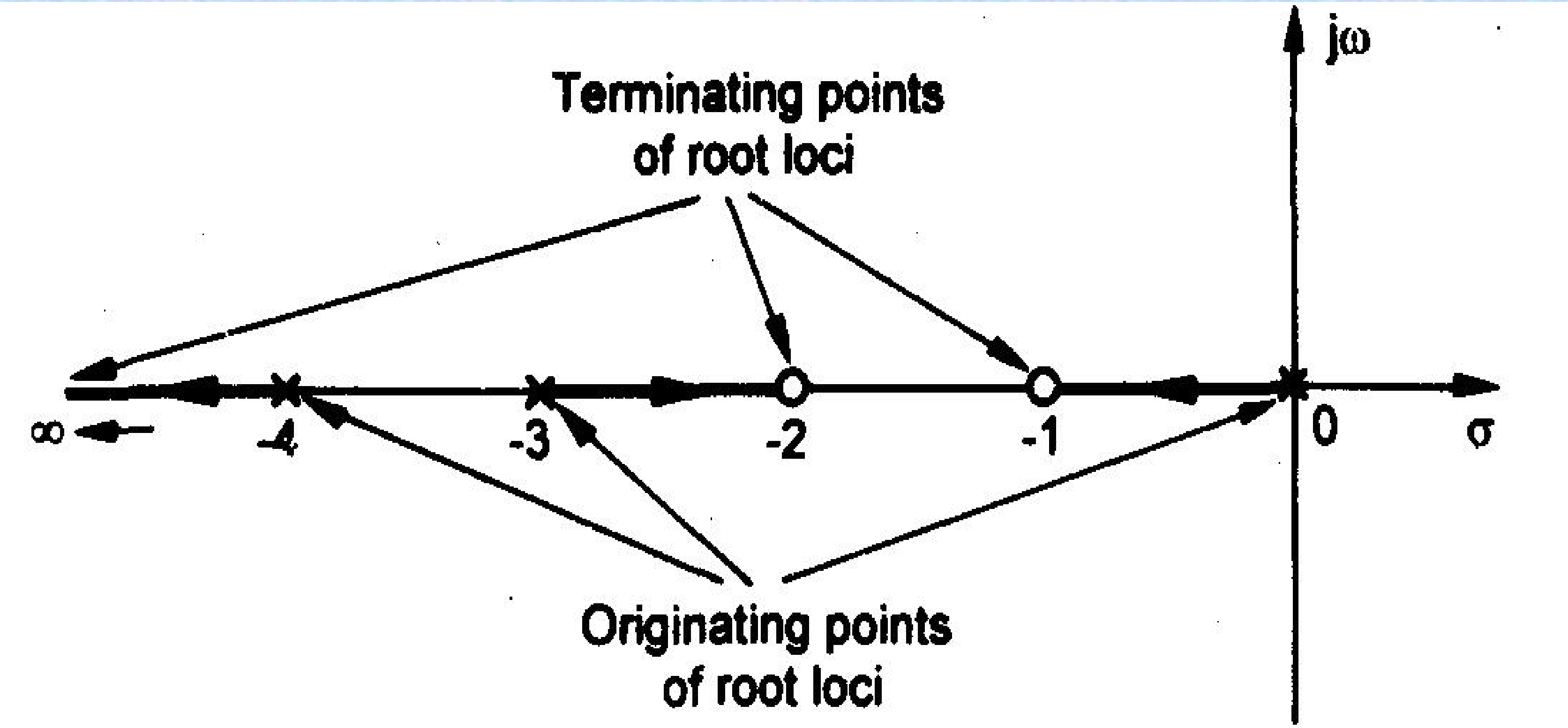


Example

Consider the system shown in Fig with the open-loop transfer function:

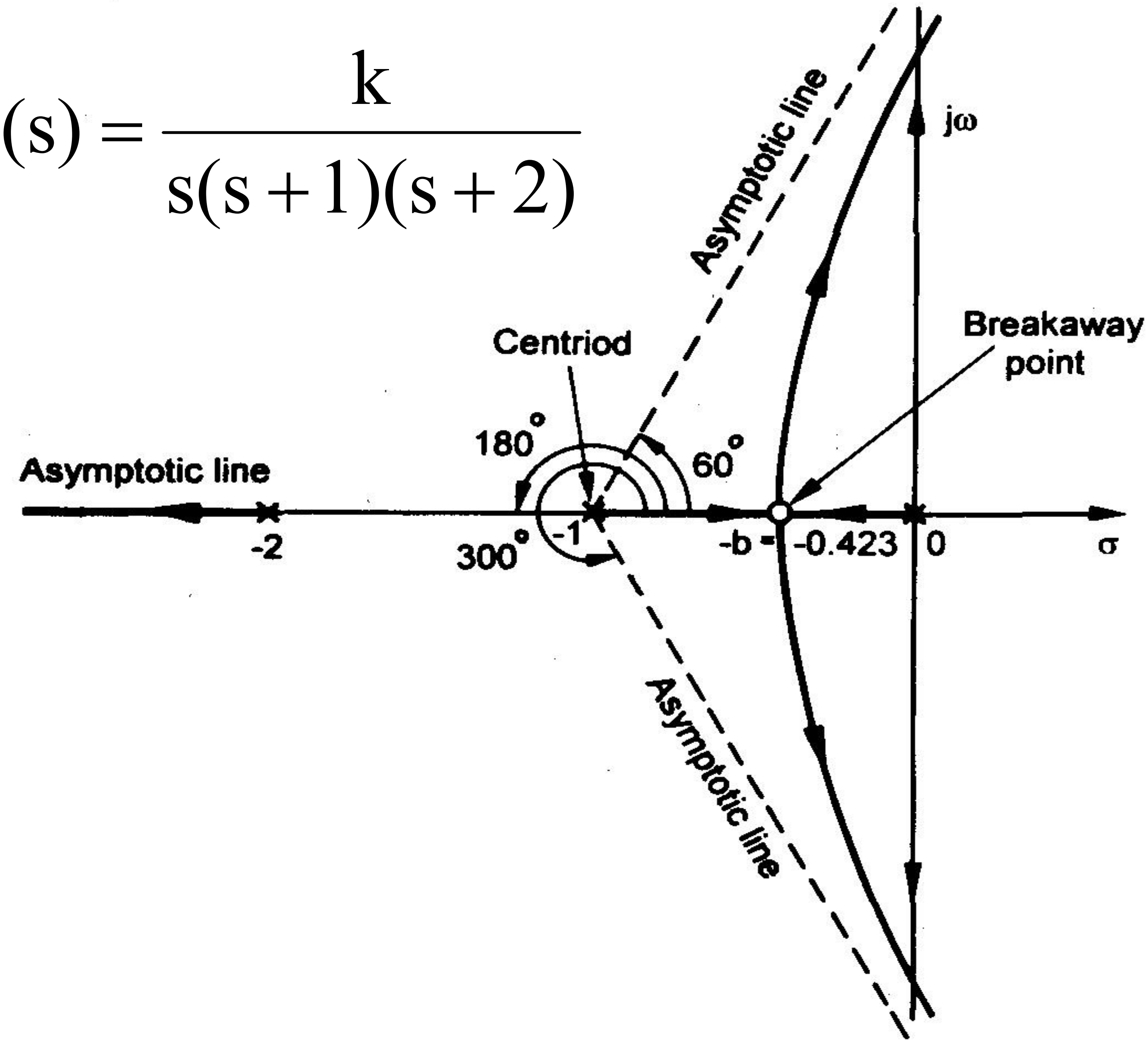


- The poles of the OLTF are at
- {s = 0, s = -3, s = -4} and
- zeros at {s = -1, s = -2}
- The root locus diagram is shown in Fig.



- 4 ● **Asymptotes of root loci**

$$G(s)H(s) = \frac{k}{s(s+1)(s+2)}$$



4 • Asymptotes of root loci

Asymptotes are the guidelines (straight lines) for branches approaching to infinity.

- **Number of asymptotes = n-m**
- **Angles of the asymptotes are given by:**

$$\theta_i = \frac{(2i+1)}{n-m} \times 180^o$$

N ≠ m where i = 0,1,2,3,....

- ● **Intersection of the asymptotes (Centroid):**
- **The intersection of the asymptotes with the real axis is given by:**

$$\sigma = \frac{\sum \text{real parts of poles} - \sum \text{real parts of zeros}}{n - m}$$

- **σ is the centroid and is always a real number.**

- **illustrative Example**
- **For the transfer function**

$$G(s)H(s) = \frac{K}{(s+1)(s^2+4s+8)}$$

- **Calculate angles of asymptotes and the centroid.**

- **Solution:**
- ● No. of poles n=3, No. of zeros m=0
- ● Since *n-m* = 3-0 =3, there are 3 branches terminate at $s=\infty$
- ● No. of asymptotes = 3.
- ●The poles of G(s)H(s) when K=0 are:
- at { s = -1, s = -2+j2, s = -2-j2 }

The angles of asymptotes are:

$$i = 0 : \theta_0 = \frac{180}{3} = 60^o$$

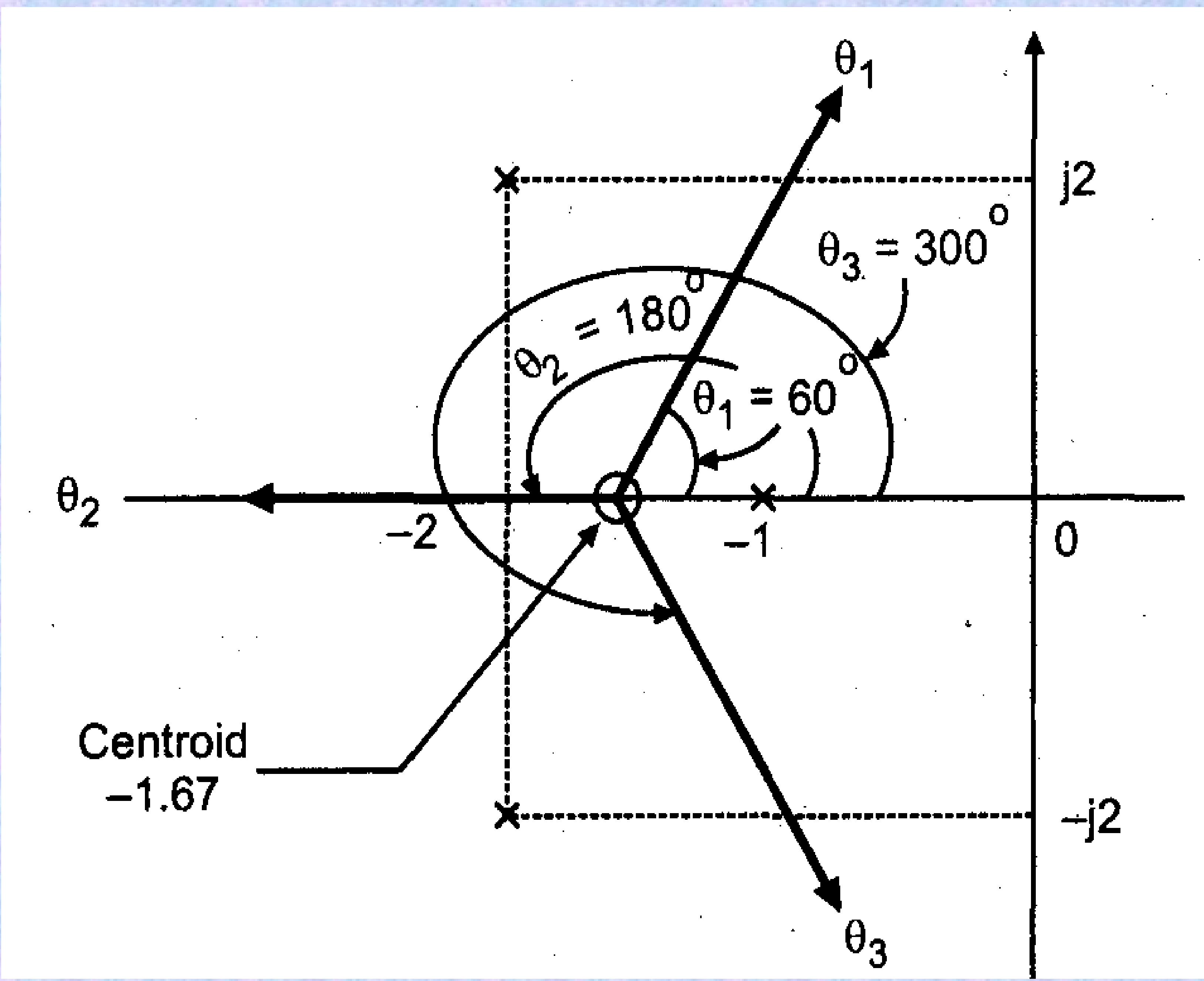
$$i = 1 : \theta_1 = \frac{540}{3} = 180^o$$

$$i = 2 : \theta_2 = \frac{900}{3} = 300^o$$

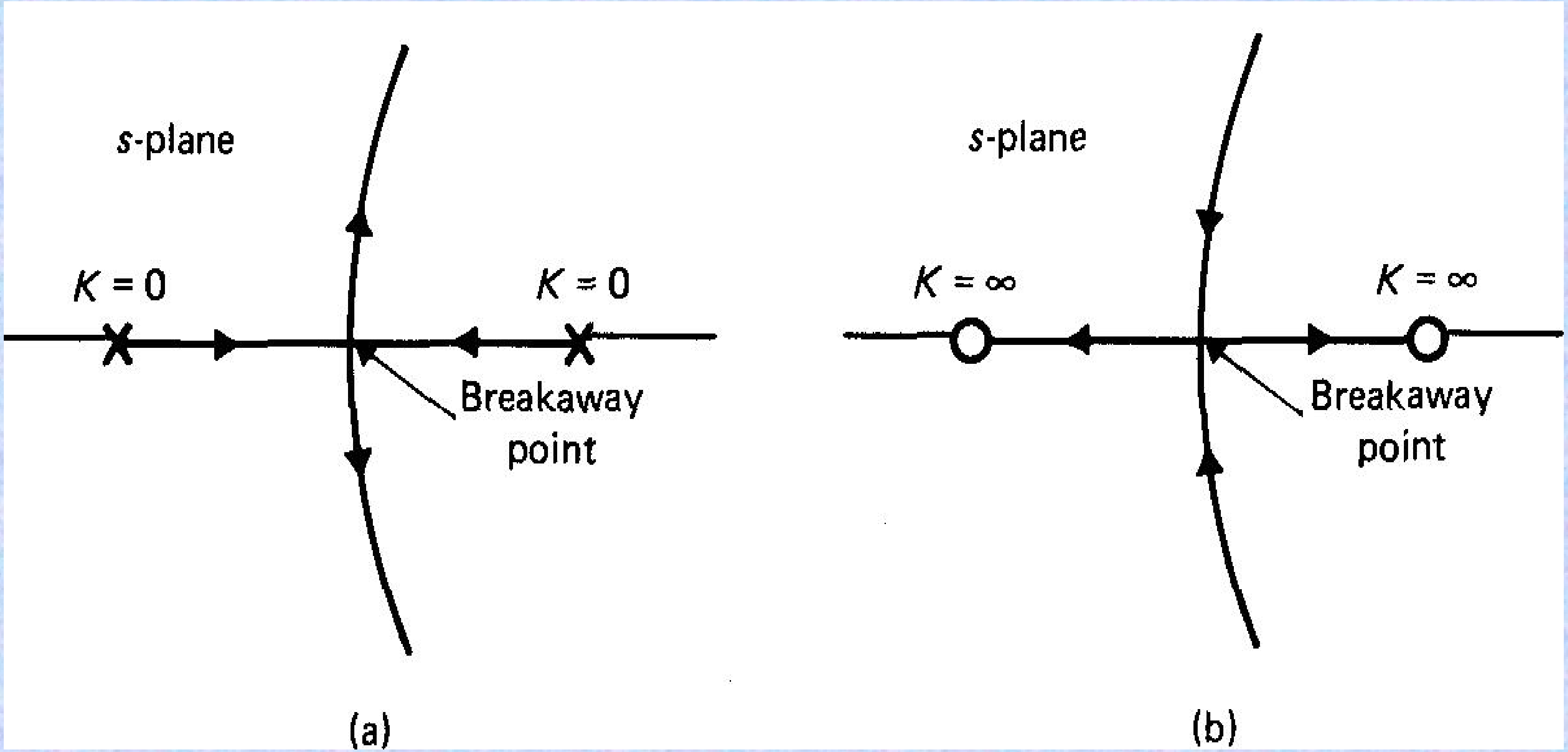
- The intersection of the asymptotes is:

$$\sigma = \frac{\sum \text{real parts of poles} - \sum \text{real parts of zeros}}{\text{n - m}}$$

$$\sigma = \frac{(0-4-1-1)-(-1)}{4-1} = -\frac{5}{3}$$


θ1
j2
θ3 = 300°
θ2 = 180°
θ1 = 60°
θ2
−2
−1
0
Centroid
−1.67
−j2
θ3

- <u>5. Breakaway (break-in) points:</u>
- **The figure illustrates the breakaway point on the real axis**



- **Determination of breakaway point**
- The breakaway point (if any) on the
- real axis can be evaluated as:
- 1. write the characteristic equation

$$1+G(s)H(s)=0$$

- 2. Separate the terms involving K and write K= f(s)
- 3.Find $dK / ds = 0$.
- 4. The roots of $dK/ds=0$ gives the breakaway points.

- **Example**
- **For the transfer function of the pervious example**

$$G(s)H(s) = \frac{k}{s(s+1)(s+2)}$$

- **Find the breakaway point?**
- **Solution:**
- **The characteristic equation is 1+G(s)H(s) =0**

$$1 + \frac{k}{s(s+1)(s+2)} = 0$$

$$s(s+1)(s+2)+K=0$$

$$K=-s(s+1)(s+2)=-(s^3+3s^2+2s)$$

$$\frac{dK}{ds}=-(3s^2+6s+2s)=0$$

- The roots of the equation dK/ds =0 are:

$$s_{1,2} = -\frac{-6 \pm \sqrt{36+24}}{6} = -0.423, \quad -1.577$$

- Since the breakaway point must lie
- between 0 and -1, it is clear that
- s = -0.423 corresponding to actual
- breakaway point.

- **6. Intersection of the root loci with the imaginary axis**
- **Determine the point at which the locus**
- **crosses the imaginary axis (if it does**
- **so) and the corresponding values of K**
- **using:**
- **1. the Routh-Hurwitz criterion or**
- **2. Replacing s by jω in the characteristic equation.**

- **Example :**
- **The characteristic equation of example is 1+G(s)H(s) =0**

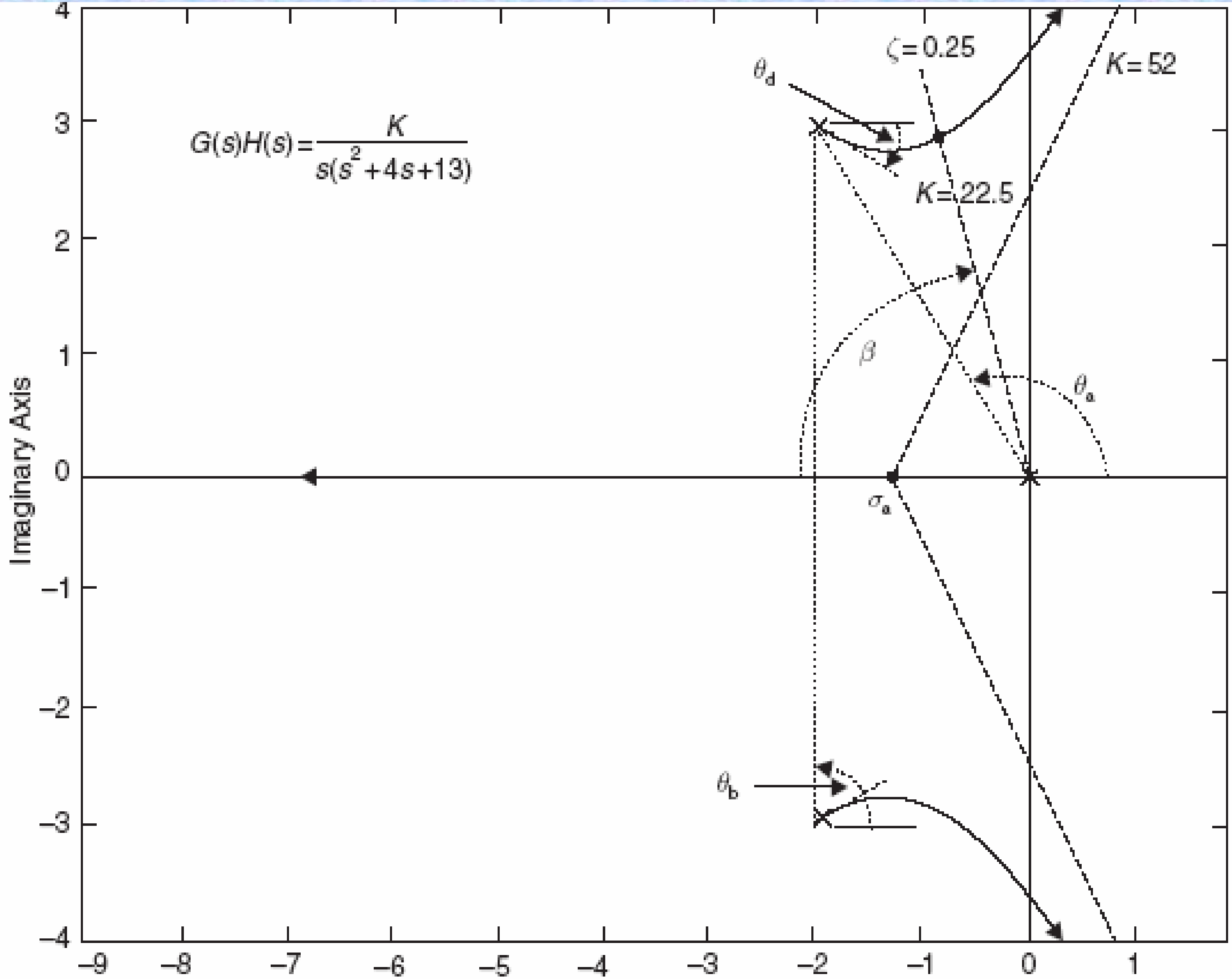
$$1+\frac{k}{s(s+1)(s+2)}=0$$

$$s(s+1)(s+2)+K=0$$

$$s^3+3s^2+2s+K=0$$

- **Application of the Routh criterion to the above equation**
- **gives the critical value of K**
- **corresponds to the location of the roots on the jω axis**
- **(locus crosses the imaginary axis).**

- **7. Angles of departure and angles of arrival of the root loci**



$G(s)H(s) = \frac{K}{s(s^2+4s+13)}$
θ_d
$\zeta = 0.25$
$K = 52$
$K = 22.5$
β
θ_a
σ_a
θ_b
Imaginary Axis
4
3
2
1
0
−1
−2
−3
−4
−9
−8
−7
−6
−5
−4
−3
−2
−1
0
1

- **7. Angles of departure and angles of arrival of the root loci**
- **The angle of departure of the locus from a pole and**
- **the angle of arrival of the locus at a zero**
- **can be determined using the phase angle criterion.**

Assume the angle of departure from a complex pole = θ_d

The angle θ_d is:

$$\theta_d = 180 - \sum \angle \text{of other poles} + \sum \angle \text{of all zeros}$$

- **Assume the angle of arrival to a complex zero = φ_a**

$$\varphi_{\mathrm{a}} = 180^{\circ} - \sum \angle \text{of other zeros} + \sum \angle \text{of all poles}$$